# Cooperative Localization and Map Building for Multi-Robot Systems: a Set Membership Approach

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**Abstract.** The problem of simultaneous localization and map building for a team of cooperating robots moving in an unknown environment is addressed. The robots have to estimate the position of static landmarks, and then localize themselves with respect to other robots and landmarks, exploiting distance and angle measurements. A novel set theoretic approach to this problem is presented. The proposed localization algorithm provides position estimates and guaranteed uncertainty regions for all robots and landmarks in the environment.

## 1 Introduction

Self localization of autonomous agents has been considered a fundamental problem in mobile robotics since long time. The problem has been tackled in several different frameworks and a number of efficient techniques are now available (see e.g. the books [1, 2, 3] and references therein). More recently, driven by the interest in planetary exploration, researches have turned their attention towards localization of vehicles moving in hostile unknown environments [4, 5]. In these cases, a map of the environment is not available and hence the more complex Simultaneous Localization And Mapping (SLAM) problem must be faced. The robot has to collect information about the environment through its sensors, and at the same time it must localize itself with respect to the map it is building. Several solutions to this problem have been proposed in the literature (see e.g. [6, 7, 8]), using different environment representations, sensor equipments, etc. However, most of these works concern the case of a single robot exploring an environment through detection of static features.

In this paper, the SLAM problem for a team of cooperative robots is addressed. This problem has attracted more and more attention in recent years, due to the enormous potentials of multi-robot exploration of unknown environments (see the special issues [9, 10]). As far as localization of multi-robot systems is concerned, most available approaches are based on probabilistic assumptions, leading to solutions that employ Extended Kalman Filters [11] or Markov localization techniques [12]. In this paper, a different approach is taken, based on the assumption that the errors affecting odometers and exteroceptive sensors are unknown-but-bounded. This naturally leads to a set theoretic formulation of the SLAM problem, similar to that given in [13] for the case of a single robot. Exploiting set approximation techniques developed in the context of set membership estimation theory (see e.g. [14, 15]) and the specific structure of the SLAM problem, efficient recursive algorithms can be devised which are suitable for real-time implementation. Here, this approach is applied to the multi-robot case, showing how set membership techniques allows for efficient information fusion, map merging and dynamic estimation of a multi-robot system evolution.

The paper is organized as follows. Section 2 presents the set membership approach to the SLAM problem for a single robot. Section 3 describes the proposed technique for solving the dynamic SLAM problem, for a team of cooperating robots. In Section 4, the fusion of set-valued maps is tackled, as a starting point for the dynamic localization of a team of robots. In Section 5, simulation experiments are reported, showing the effectiveness of the proposed approach. Concluding remarks are given in Section 6.

## 2 Set membership SLAM for a single robot

In this section, the basic features of the set membership approach to the single-robot SLAM problem are reviewed (see [13] for more details). Let us assume that a single vehicle is moving in an environment which can

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be adequately represented by a two-dimensional reference system (*flat* landscape assumption). The vehicle is equipped with odometers, providing direct measurements of the robot displacement every $T_s$ seconds. Under the assumption of slow dynamics, the robot motion can be roughly described by the simplified model

$$\xi(k+1) = \xi(k) + u(k) + G(k)w(k) \tag{1}$$

where $\xi(k) = [x(k)\ y(k)\ \theta(k)]'$ is the *pose* (position $p(k) = [x(k)\ y(k)]'$ and orientation $\theta(k)$) of the robot at time $kT_s$, $u(k)$ is the vector of x-, y- and $\theta$-displacement measurements provided by the odometric sensors, and $w(k)$ are the errors affecting these measurements (possibly shaped by a matrix $G(k)$). More complicated motion models can be considered without affecting the approach presented in the paper (see [16]).

Other common proprioceptive sensors are rate gyros, accelerometers and sun sensors, that are used for orientation determination [5]: they usually provide a noisy measurement of the robot orientation with respect to a fixed absolute direction, such as

$$\Theta(k) = \theta(k) + v_\theta(k), \tag{2}$$

where $v_\theta(k)$ is the noise affecting the absolute orientation measurement.

In order to localize itself, the robot has to build a reliable description of the environment, because no a priori map is available. To do this, the robot selects distinguishable landmarks $L_i = [x_{L_i}\ y_{L_i}]$, $i = 1, \ldots, n$ in the environment and performs measurements with respect to these landmarks. Exteroceptive measurements are used, together with the robot dynamic model, to simultaneously estimate both robot and landmarks positions at each time $k$: this is the aim of the SLAM problem. The problem can be formulated as the state estimation of a dynamic system, whose state vector is given by $X(k) = [\xi'(k)\ L'_1\ \ldots\ L'_n]' \in \mathbb{R}^{3+2n}$. As the selected landmarks are static ones, the state update equation is

$$X(k+1) = X(k) + E_3 u(k) + E_3 G(k) w(k) \tag{3}$$

where $E_3 = [I_3\ 0]' \in \mathbb{R}^{(3+2n)\times 3}$ and $I_3$ is the $3 \times 3$ identity matrix.

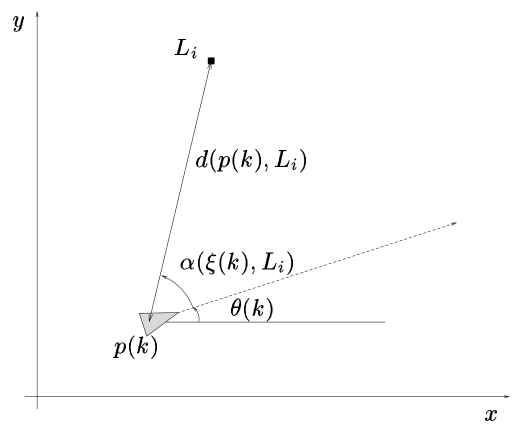



Fig. 1. Distance and heading measurements.

It is assumed that two sets of measurements are taken by the robot: (i) distance from a landmark; (ii) angle between robot orientation and the direction of a landmark. These measurements can be obtained by several kind of sensors: millimeter-wave radars [7], laser range finders [11], stereovision systems [17, 18]. Measurement equations take on the form

$$\begin{aligned} \Delta_i(k) &= d_i(X(k)) + v_{d_i}(k) \\ A_i(k) &= \alpha_i(X(k)) + v_{\alpha_i}(k) \end{aligned} \tag{4}$$

where $\Delta_i(k)$ and $A_i(k)$ are the actual readings provided by the sensors at time $k$; $v_{d_i}(k)$ and $v_{\alpha_i}(k)$ are measurement noise affecting, respectively, the distance and the heading measurements; and

$$\begin{aligned} d_i(X(k)) &= d(p(k), L_i) \triangleq \sqrt{(x(k) - x_{L_i})^2 + (y(k) - y_{L_i})^2}\ , \\ \alpha_i(X(k)) &= \alpha(\xi(k), L_i) \triangleq \operatorname{arctan}_2\{y_{L_i} - y(k), x_{L_i} - x(k)\} - \theta(k)\ , \end{aligned} \tag{5}$$

with $\arctan_2\{b, a\}$ denoting the four quadrant inverse tangent (see Fig. 1).

According to (3)-(5), dynamic estimation of the state vector $X(k)$ can be tackled via Extended Kalman Filter techniques [7, 8], provided that statistical properties of disturbances $w(k)$, $v_\theta(k)$, $v_{d_i}(k)$ and $v_{\alpha_i}(k)$ are a priori known or can be reliably estimated. Here a different approach is taken, which relies on the only hypothesis that the above disturbances are unknown-but-bounded, i.e.

$$\|w(k)\|_\infty \leq \epsilon^w(k), \quad |v_\theta(k)| \leq \epsilon^{v_\theta}(k), \tag{6}$$

$$|v_{d_i}(k)| \leq \epsilon^{v_d}(k), \quad |v_{\alpha_i}(k)| \leq \epsilon^{v_\alpha}(k), \tag{7}$$

where $\epsilon^w(k)$, $\epsilon^{v_\theta}(k)$, $\epsilon^{v_d}(k)$ and $\epsilon^{v_\alpha}(k)$ are known positive scalars $\|v\|_\infty = \max_i |v_i|$ The above bounds need not be the same for different measurements $d_i$ and $\alpha_i$ and for different components of vector $w(k)$. This feature will not be considered in the treatment of the paper, the extension being straightforward.

Assumptions (6)-(7) lead to the notion of feasible state vectors. Given sensor readings (4), the feasible states are those compatible with the given measurements, i.e. the states belonging to the *measurement set*

$$\mathcal{M}(k) = \mathcal{M}_o(k) \cap \left[\cap_{i=1}^n \mathcal{M}_i(k)\right], \tag{8}$$

where

$$\mathcal{M}_o(k) = \{X : \quad |\Theta(k) - \theta| \leq \epsilon^{v_\theta}(k)\} \tag{9}$$

$$\mathcal{M}_i(k) = \{X : \quad |\Delta_i(k) - d_i(X)| \leq \epsilon^{v_d}(k) \text{ and } |A_i(k) - \alpha_i(X)| \leq \epsilon^{v_\alpha}(k)\}. \tag{10}$$

As a direct consequence, the dynamic SLAM problem can be formulated in the following set-theoretic form.

*Set Membership SLAM Problem (SM-SLAM)*: Let $\Xi(0) \subset \mathbb{R}^{3+2n}$ be a set containing the initial vehicle pose and landmarks position $X(0)$. Given the dynamic model (3) and the measurement equations (4)-(5), find at each time $k = 1, 2, \ldots$, the *feasible state set* $\Xi(k|k)$, which is the set of state vectors $X(k)$ compatible with robot dynamics, assumptions (6)-(7) and measurements collected up to time $k$.

The solution of the SM-SLAM problem is given by the set-valued recursion

$$\Xi(0|0) = \Xi(0), \tag{11}$$

$$\Xi(k|k-1) = \Xi(k-1|k-1) + E_3 u(k-1) + E_3 G(k-1)\epsilon^w(k-1)\mathcal{B}_\infty, \tag{12}$$

$$\Xi(k|k) = \Xi(k|k-1) \bigcap \mathcal{M}(k), \tag{13}$$

where $\mathcal{B}_\infty$ is the unit ball in the $\ell_\infty$ norm. Algebraic operators in (11)-(13) are to be intended as set operators. It is important to notice that distance measurements (i.e. $\Delta_i(k)$ in eq. (4)) are relative. Therefore the origin of the reference system can be chosen arbitrarily: without loss of generality, the origin is picked as the robot initial position. The reference axis for robot orientation, it can be chosen according to the values initially provided by the proprioceptive sensors measuring the robot heading.

The main property of recursion (11)-(13) is to provide, at each time step, all the state values that are compatible with all the available information: the true state is guaranteed to belong to sets $\Xi(k|k)$, and the size of such sets give a measure of the quality of the estimates. Unfortunately, the exact computation of the sets $\Xi$ in (12)-(13) is generally a prohibitive task, because the measurement set $\mathcal{M}(k)$ is the intersection of nonlinear and nonconvex sets. Therefore, set approximations must be pursued. In [13, 16], efficient techniques for computing guaranteed outer approximations of the feasible sets have been proposed. The basic idea is as follows: choose a class of simple approximating regions $\mathcal{R}$ (e.g. axis-aligned boxes, parallelotopes, ellipsoids, etc.) and select at each time $k$ an element $\mathcal{R}(k|k)$ in the class, such that $\Xi(k|k) \subset \mathcal{R}(k|k)$. The selection of $\mathcal{R}(k|k)$ involves a crucial trade-off. On one hand, the "size" of the approximating region (typically, the area of feasible position sets for robot and landmarks) must be minimized in order to reduce conservativeness. On the other side, the computational complexity must be kept small, in order to obtain approximation algorithms suitable for on-line implementation.

## 3 Multi-robot SLAM

In this section, the above formulation will be extended to the case of $m$ robots, whose poses at time $k$ will be denoted by $p_1(k), \ldots, p_m(k)$, moving in an environment containing $n$ static landmarks $L_1, \ldots, L_n$. The state vector to be estimated becomes $X(k) = [\xi_1'(k) \ \ldots \ \xi_m'(k) \ L_1' \ \ldots \ L_n']' \in \mathbb{R}^{3m+2n}$. Assuming robot dynamics $\xi_i(k+1) = \xi_i(k) + u_i(k) + G_i(k)w_i(k)$ for each agent, the state update equation turns out to be

$X(k+1) = X(k) + E_m u(k) + E_m G(k) w(k)$ where $E_m = [I_{3m}\ 0]' \in \mathbb{R}^{(3m+2n)\times 3m}$; $G(k)$ is the block-diagonal matrix of blocks $G_i(k)$, $i = 1, \dots, m$; $u(k) = [u_1'(k) \dots u_m'(k)]'$ and $w(k) = [w_1'(k) \dots w_m'(k)]'$. Absolute proprioceptive sensors will provide the orientation measurements $\Theta_i(k) = \theta_i(k) + v_{\theta_i}(k)\quad i = 1.\dots, m$.

Since each agent performs measurements with respect to all visible features in the environment (the other robots and the static landmarks), the maximum amount of information provided by exteroceptive sensors is

$$\Delta_i^j(k) = d_i^j(X(k)) + v_{d_i^j}(k), \quad A_i^j(k) = \alpha_i^j(X(k)) + v_{\alpha_i^j}(k)$$

for $j = 1, \dots, m$, $i = 1, \dots, m+n$, $i \neq j$, where $\Delta_i^j(k), A_i^j(k)$ are the actual readings of the $j$-th robot sensors towards the $i$-th feature in the environment (with the notation $d_i^j(X) = d(p_j, p_i)$, $i = 1, \dots, m$ and $d_{i+m}^j(X) = d(p_j, L_i)$, $i = 1, \dots, n$, and similarly for $\alpha_i^j$). In the following we will denote by $\Omega_i^j(k)$ the pair of measurements $\Delta_i^j(k), A_i^j(k)$.

Similarly to what has been done in the single-robot case, unmodeled dynamics errors and measurement noises are assumed to be unknown-but-bounded, as in (6)-(7). Then, the measurement sets $\mathcal{M}^j(k)$, $\mathcal{M}_o^j(k)$ and $\mathcal{M}_i^j(k)$ associated to $\Omega_i^j(k)$, can be defined as in the single robot case

$$\mathcal{M}^j(k) = \mathcal{M}_o^j(k) \cap \left[\cap_{i=1}^n \mathcal{M}_i^j(k)\right] \tag{14}$$

$$\mathcal{M}_o^j(k) = \{X : |\Theta_j(k) - \theta_j| < \epsilon^{v_\theta}\} \tag{15}$$

$$\mathcal{M}_i^j(k) = \left\{X : |\Delta_i^j(k) - d_i^j(X)| < \epsilon^{v_d}(k) \text{ and } |A_i^j(k) - \alpha_i^j(X)| < \epsilon^{v_\alpha}(k)\right\}. \tag{16}$$

The exact feasible state set is obtained (at least in principle) through a recursive procedure similar to (11)-(13). Clearly, the exact computation of sets $\Xi$ cannot be tackled in practice. Therefore, an efficient set approximation strategy exploiting the specific structure of the multi-robot SM-SLAM is needed. The main idea is to decompose the approximation of the feasible set $\Xi$ into $m+n$ approximations of 2D feasible subsets for the position of each feature in the environment, plus $m$ interval approximations for the feasible orientation of each robot.

Let $\mathcal{R}_{p_j}$ and $\mathcal{R}_{L_h}$ denote outer approximations of the feasible position sets of robot $p_j$ and landmark $L_h$ respectively, chosen in the set class $\mathcal{R}$. Moreover, let $\mathcal{R}_{o_j}$ denote outer approximation of the feasible orientation interval of the $j$-th robot. The sets defined below will be useful in the following treatment:

- $\mathcal{M}_{p_i}(\Omega_j^i, \mathcal{R}_{p_j}, \mathcal{R}_{o_i})$: set of robot $p_i$ positions, compatible with sets $\mathcal{R}_{p_j}$, $\mathcal{R}_{o_i}$ and measurements $\Omega_j^i$;
- $\mathcal{M}_{p_i}(\Omega_{h+m}^i, \mathcal{R}_{L_h}, \mathcal{R}_{o_i})$: set of robot $p_i$ positions, compatible with sets $\mathcal{R}_{L_h}$, $\mathcal{R}_{o_i}$ and measurements $\Omega_{h+m}^i$;
- $\mathcal{M}_{p_i}(\Omega_i^j, \mathcal{R}_{p_j}, \mathcal{R}_{o_j})$: set of robot $p_i$ positions, compatible with sets $\mathcal{R}_{p_j}$, $\mathcal{R}_{o_j}$ and measurements $\Omega_i^j$;
- $\mathcal{M}_{L_h}(\Omega_{h+m}^j, \mathcal{R}_{p_j}, \mathcal{R}_{o_j})$: set of landmark $L_h$ positions, compatible with sets $\mathcal{R}_{p_j}$, $\mathcal{R}_{o_j}$ and meas. $\Omega_{h+m}^j$;
- $\mathcal{M}_{o_i}(\Omega_j^i, \mathcal{R}_{p_j}, \mathcal{R}_{p_i})$: set of robot $p_i$ orientations, compatible with sets $\mathcal{R}_{p_j}$, $\mathcal{R}_{p_i}$ and measurements $\Omega_j^i$;
- $\mathcal{M}_{o_i}(\Omega_{h+m}^i, \mathcal{R}_{L_h}, \mathcal{R}_{p_i})$ set of robot $p_i$, orientations compatible with sets $\mathcal{R}_{L_h}$, $\mathcal{R}_{p_i}$ and meas. $\Omega_{h+m}^i$.

Notice that the first two sets show how the uncertainty on the measured features (robots and landmarks) affects the uncertainty on the robot performing those measurements. On the contrary, the third and fourth sets state how the uncertainty on the robot performing the measurements affects that of the measured features. Concerning the last two sets, notice that each robot can not perform measurements on other agents orientation; consequently, information on each robot orientation can be derived only by the measurements performed by the robot itself on other robots and on static landmarks.

For the pair of measurements $\Omega_i^j$ considered in this paper, the 2D approximate position sets considered above turn out to be the sum of sectors of circular coronas and the sets $\mathcal{R}_{p_j}$ or $\mathcal{R}_{L_h}$. In particular, if axis-aligned boxes are used as approximating regions, it is easy to compute the minimum area box containing one of the above sets (see [13] for details). For example, Fig. 2 shows how $\mathcal{M}_{L_h}$ depends on measurements $\Omega_{h+m}^j$, robot orientation uncertainty set $\mathcal{R}_{o_j}$ and robot position uncertainty set $\mathcal{R}_{p_j}$.

Let $\overline{\mathcal{R}}\{\mathcal{S}\}$ denote the minimum area set in the class $\mathcal{R}$, containing the set $\mathcal{S}$. Hence, the following suboptimal recursive approximation strategy provides a solution to the multi-robot SM-SLAM problem.

**Step 0 (Initialization)**. Let $\mathcal{R}_{p_i}(0|0)$, $\mathcal{R}_{L_j}(0|0)$ be 2D sets containing the projection of $\Xi(0|0)$ on the subspaces spanned respectively by $p_i$ and $L_j$, $i = 1, \dots, m$, $j = 1, \dots, n$. Let $\mathcal{R}_{o_i}(0|0)$ be the interval obtained projecting $\Xi(0|0)$ on $\theta_i$, $i = 1, \dots, m$.

For $k = 1, 2, \dots$ repeat the following steps:

**Step 1**. Time update of robot uncertainty sets. For $i = 1, \ldots, m$:

$$\mathcal{R}_{p_i}(k|k-1) \otimes \mathcal{R}_{o_i}(k|k-1) = \overline{\mathcal{R}}\left\{\mathcal{R}_{p_i}(k-1|k-1) \otimes \mathcal{R}_{o_i}(k-1|k-1) + u_i(k-1) + G_i(k-1)\epsilon^{w_i}(k-1)\mathcal{B}_\infty\right\}$$

where $\otimes$ denotes Cartesian product between sets.
**Step 2**. Robot orientation update (based on orientation proprioceptive sensors).

$$\widehat{\mathcal{R}}_{o_i}(k|k) = \mathcal{R}_{o_i}(k|k-1) \cap \mathcal{M}_o^i(k).$$

**Step 3**. Robot position update (based on measurements performed by the same robot). For $i = 1, \ldots, m$ :
- let

$$\mathcal{A}_{1,i} = \bigcap_{\substack{j=1\\ j\neq i}}^{m} \overline{\mathcal{R}}\left\{\mathcal{M}_{p_i}(\Omega_j^i(k), \mathcal{R}_{p_j}(k|k-1), \widehat{\mathcal{R}}_{o_i}(k|k))\right\},$$

$$\mathcal{A}_2 = \bigcap_{h=1}^{n} \overline{\mathcal{R}}\left\{\mathcal{M}_{p_i}(\Omega_{h+m}^i(k), \mathcal{R}_{L_h}(k-1|k-1), \widehat{\mathcal{R}}_{o_i}(k|k))\right\};$$

- compute

$$\widehat{\mathcal{R}}_{p_i}(k|k) = \overline{\mathcal{R}}\left\{\mathcal{R}_{p_i}(k|k-1) \bigcap \mathcal{A}_{1,i} \bigcap \mathcal{A}_2\right\}.$$

The set $\mathcal{A}_{1,i}$ contains the feasible set for agent $p_i$ based on other robots uncertainty and measurements performed on them by the same agent $p_i$. Set $\mathcal{A}_2$ contains the feasible set for robot $p_i$ based on landmarks uncertainty and measurements performed by the same robot.
**Step 4**. Robot position update (based on measurements performed by other robots). For $i = 1, \ldots, m$ :
- let

$$\mathcal{A}_{3,i} = \bigcap_{\substack{j=1\\ j\neq i}}^{m} \overline{\mathcal{R}}\left\{\mathcal{M}_{p_i}(\Omega_i^j(k), \widehat{\mathcal{R}}_{p_j}(k|k), \widehat{\mathcal{R}}_{o_j}(k|k))\right\};$$

- compute

$$\mathcal{R}_{p_i}(k|k) = \overline{\mathcal{R}}\left\{\widehat{\mathcal{R}}_{p_i}(k|k) \bigcap \mathcal{A}_{3,i}\right\}.$$

Set $\mathcal{A}_{3,i}$ contains the feasible position set for robot $p_i$ depending on other agents current uncertainty (updated in step 2) and their measurements performed on robot $p_i$.
**Step 5**. Landmark measurement update. For $h = 1, \ldots, n$ :
- let

$$\mathcal{A}_4 = \bigcap_{j=1}^{m} \overline{\mathcal{R}}\left\{\mathcal{M}_{L_h}(\Omega_{h+m}^j(k), \mathcal{R}_{p_j}(k|k), \widehat{\mathcal{R}}_{o_j}(k|k))\right\};$$

- compute

$$\mathcal{R}_{L_h}(k|k) = \overline{\mathcal{R}}\left\{\mathcal{R}_{L_h}(k-1|k-1) \bigcap \mathcal{A}_4\right\}.$$

Set $\mathcal{A}_4$ contains the feasible set for landmark $L_h$ based on robots current uncertainty (updated in step 4) and measurements performed by all robots on the same landmark $L_h$.
**Step 6**. Robot orientation update (using measurements performed by the same robot). For $i = 1, \ldots, m$ :
- let

$$\mathcal{A}_{5,i} = \bigcap_{\substack{j=1\\ j\neq i}}^{m} \mathcal{M}_{o_i}\left(\Omega_j^i(k), \mathcal{R}_{p_j}(k|k), \mathcal{R}_{p_i}(k|k)\right);$$

$$\mathcal{A}_6 = \bigcap_{h=1}^{n} \mathcal{M}_{o_i}\left(\Omega_{h+m}^i(k), \mathcal{R}_{L_h}(k|k), \mathcal{R}_{p_i}(k|k)\right);$$

- compute

$$\mathcal{R}_{o_i}(k|k) = \widehat{\mathcal{R}}_{o_i}(k|k) \cap \mathcal{A}_{5,i} \cap \mathcal{A}_6.$$

Interval $\mathcal{A}_{5,i}$ contains the feasible orientations of the $i$-th robot, based on other robot uncertainties and measurements performed on them by the same $i$-th agent. Interval $\mathcal{A}_6$ contains the feasible orientations of the $i$-th robot based on landmarks uncertainty and measurements performed by the same robot.

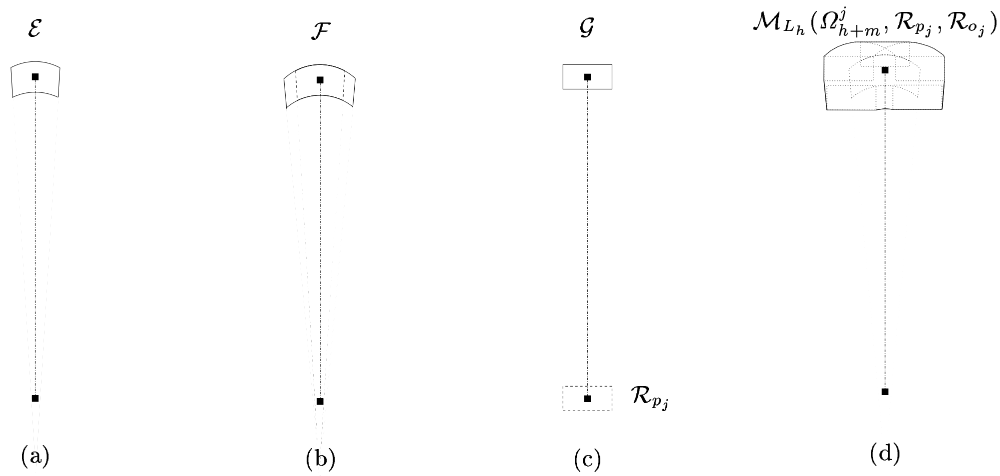



Fig. 2. Dependence of the set $\mathcal{M}_{L_h}$ on sets $\mathcal{R}_{p_j}$, $\mathcal{R}_{o_j}$ and on measurements $\Omega^j_{h+m}$. (a) Set $\mathcal{E}$ is the feasible landmark set compatible with measurements $\Omega^j_{h+m}$, nominal robot position $\mathrm{cen}(\mathcal{R}_{p_j})$ and nominal robot orientation $\mathrm{cen}(\mathcal{R}_{o_j})$ (where $\mathrm{cen}(\mathcal{X})$ denotes the center of set $\mathcal{X}$); (b) Set $\mathcal{F}$ is the feasible landmark set compatible with measurements $\Omega^j_{h+m}$, the robot orientation set $\mathcal{R}_{o_j}$ and nominal robot position $\mathrm{cen}(\mathcal{R}_{p_j})$ (set $\mathcal{E}$ is dashed); (c) Set $\mathcal{G}$ is the feasible landmark set compatible with nominal measurements $\Omega^j_{h+m}$, nominal robot orientation $\mathrm{cen}(\mathcal{R}_{o_j})$ and robot position uncertainty set $\mathcal{R}_{p_j}$ (dashed); (d) Construction of set $\mathcal{M}_{L_h}(\Omega^j_{h+m}, \mathcal{R}_{p_j}, \mathcal{R}_{o_j}) = \mathcal{F} + \mathcal{G}$.

Clearly, in the above strategy there is a redundancy, due to the fact that the same measurements are processed several times, in Step 3 to 6. This redundancy is useful because in each step set approximations are involved and hence measurement reprocessing leads to approximation refinement. In this respect, a further uncertainty reduction can be obtained by repeating iteratively Steps 3-6 (heuristic strategies can be easily formulated to properly choose the number of iterations, on the basis of a trade-off between uncertainty reduction and computational burden). Notice that when axis-aligned boxes are considered, the intersections in the sets $\mathcal{A}$ are simple 2D box intersections, and also the minimum box approximations $\overline{\mathcal{R}}\{\cdot\}$ are very easy to compute. More sophisticated set approximations can be considered (e.g., parallelotopes) at the price of a higher computational complexity.

It is worth remarking that, no matter which class of approximating sets $\mathcal{R}$ is chosen, the above strategy is conceived so that a guaranteed outer approximation of the true feasible state set $\Xi(k|k)$ is obtained at each $k$. In fact, it can be easily shown that

$$\Xi(k|k) \subset \mathcal{R}_{p_1}(k|k) \otimes \mathcal{R}_{o_1}(k|k) \otimes \cdots \otimes \mathcal{R}_{p_m}(k|k) \otimes \mathcal{R}_{o_m}(k|k) \otimes \mathcal{R}_{L_1}(k|k) \otimes \cdots \otimes \mathcal{R}_{L_n}(k|k).$$

The only problem remaining is now that of initializing the above recursive procedure. This is a crucial step, because a rough initialization would cause the whole procedure to fail. Indeed, if the initial sets $\mathcal{R}_{p_i}(0|0)$, $\mathcal{R}_{o_i}(0|0)$ and $\mathcal{R}_{L_j}(0|0)$ are too large (for example, including the whole area in which the robots are moving and all possible robot orientations) the procedure will typically provide no uncertainty reduction. This is a consequence of the simplification introduced by state decomposition, in a context where only relative measurements are available. A natural way to overcome this problem is to exploit the information that each robot is able to acquire by itself in the initialization step. In fact, every agent can solve a single-robot SM-SLAM problem in a preliminary static phase and then provide the estimated map (in its own reference system) to a central unit. The unit has to merge all the maps to obtain a unique map in an "absolute" reference system, and then send it to all the agents that will use it to initialize the recursive procedure previously outlined. The next section will address the problem of merging set-valued maps worked out by different agents.

## 4 Map fusion in the set membership framework

In the single robot SLAM problem, the initial position of the exploring agent can be fixed arbitrarily (in the robot centered reference system) and all the environment features are then estimated with respect to that initial position. When operating with multiple agents, the starting position of each robot is not known and cannot be arbitrarily fixed. As long as robot heading is concerned, in the single robot case the absolute orientation sensor provides a heading measurement, which allows one to rotate the initial 2D uncertain map onto an absolute orientation reference system (this amounts to consider $A_i(0) + \Theta(0)$ as absolute orientation measurements, corrupted by noise bounded by $\epsilon^{v_\alpha} + \epsilon^{v_\theta}$). Clearly, the same can be done by each agent in the initial step of the multi-robot case. Hence, by computing (or approximating) the initial 2D feasible position sets of each feature, based on measurements $\Omega_i^j(0)$ and $\Theta_j(0)$, the $j$-th agent is able to produce a self-centered initial map of the environment, within an absolute orientation reference system. In other words, only relative translation among the different initial maps produced by the robots is unknown [5]. Clearly, all single-robot maps are a valid, suboptimal representation of the environment. The quality of the maps will generally vary from robot to robot. While it is possible to choose the "more precise" map as a global map for all the robots, this choice does not use all the available information. Exploiting the "feasibility" property (i.e., the fact that each set-valued map contains the correct environment representation), it is possible to obtain a global refined map, by finding a description which satisfies all the constraints of each map. This problem is addressed below.

### 4.1 One-dimensional map fusion

Let us consider axis-aligned boxes as approximating sets $\mathcal{R}$. Exploiting the fact that the initial maps provided by the robots share the same orientation, it is possible to decouple the 2D map fusion into two separate 1D map fusions, which further simplifies computations. Therefore, let us consider $m$ robots and $n$ landmarks spread on a line. Each robot produces his own 1D self-centered map, containing $m+n$ intervals. Let us call $C_j^i$ the center of the set corresponding to the $j$-th feature in the map, provided by the $i$-th agent. Moreover, let us call $C_{j-}^i$ and $C_{j+}^i$ the minimum and maximum value of the interval with center $C_j^i$.
In terms of relative positions between sets, each ordered map $i$ provides a certain number of constraints, such as

$$\begin{aligned} m_j - M_k &\geq C_{j-}^i - C_{k+}^i \ , \\ M_j - m_k &\leq C_{j+}^i - C_{k-}^i \ . \end{aligned} \tag{17}$$

for $j > k$, $k \geq 1$, $j \leq m+n$, where $m_j$ and $M_j$ are respectively the minimum and maximum admissible value for the position of the $j$-th feature of the map. Note that (17) provides $(m+n)(m+n-1)$ inequalities in $2(m+n)$ unknowns for each map $i$: as a general consequence, for $m+n \geq 3$, each map provides an overconstrained system, which results to be always feasible. If all the inequalities (17) provided by all maps are considered, one has a system of $m(m+n)(m+n-1)$ inequalities. Since the SM approach guarantees that each set contains the true feature position, the latter will satisfy all the constraints in (17) for each $i$. To get the tighter constraints from (17), one can choose as "minimum" distance between set $j$ and $k$ the maximum of all minimum distances (and similarly as "maximum" distance, the minimum of all maximum distances). This leads to

$$\begin{aligned} m_j - M_k &= \max_{i=1,\dots,m}(C_{j-}^i - C_{k+}^i), \\ M_j - m_k &= \min_{i=1,\dots,m}(C_{j+}^i - C_{k-}^i). \end{aligned} \tag{18}$$

This time, however, the constraints provided by (18) will not generally be compatible, thus giving an overconstrained, unsolvable linear system. In order to get a solution $(m_k, M_k)$, $k = 1, \dots, m+n$, it is possible to relax system (18) into a Linear Programming (LP) problem, where each of the equality constraints is transformed into a suitable inequality, while the objective function can be chosen so that the global map uncertainty is minimized. This leads to the following LP problem

$$\min_{\substack{m_k, M_k \\ k=1,\dots,m+n}} \sum_{k=1,\dots,m+n} M_k - m_k \tag{19}$$

subject to the constraints

$$\begin{aligned} m_j - M_k &\leq \max_{i=1,\dots,m}(C_{j-}^i - C_{k+}^i), \quad j > k; \\ M_j - m_k &\geq \min_{i=1,\dots,m}(C_{j+}^i - C_{k-}^i), \quad j > k; \\ m_k - M_k &\leq 0, \quad k = 1, \dots, m+n; \\ m_1 &= 0. \end{aligned}$$

The last equality constraint fixes a point of the map as a common reference for all robots.

By solving problem (19), one obtains the maximum and minimum value for the uncertainty interval of each feature, and thus the map (central estimate plus interval width) merging all the information. Notice that the solution of problem (19) provides the minimum uncertainty map (in the sense defined by the chosen cost function) satisfying all the compatible constraints of equations (18), the other constraints being relaxed of the minimum quantity needed to guarantee solvability of the system.

### 4.2 Dynamic SM-SLAM initialization

As stated before, the use of boxes permits to transform the generic 2D map fusion problem into two 1D problems, by considering separately the $x$ and $y$ coordinates. Therefore, a 2D map fusion requires the solution of just two LP problems in the form (19). The resulting map can be used as the initial condition for the dynamic multi-robot SM-SLAM algorithm of Section 3. In particular, in the Step 0 of the algorithm, the approximate feasible position sets $\mathcal{R}_{p_i}(0|0)$, $\mathcal{R}_{L_j}(0|0)$ are obtained from the 2D map computed via the two LP optimizations, while the intervals $[\Theta_i(0) - \epsilon^{v_\theta}(0), \Theta_i(0) + \epsilon^{v_\theta}(0)]$ can be chosen as feasible orientation intervals $\mathcal{R}_{o_i}(0|0)$. Then, the time recursion can start, and Steps 1-6 can be repeated for each time $k$.

## 5 Simulation results

The performances of the multi-robot SLAM algorithm have been tested via numerical simulations. First, a static setting has been considered. In order to quantify the improvement provided by map fusion (performed as in Section 4) over single-robot set-valued maps, several simulations with increasing number of robots ($m \geq 2$) and non-sensing features ($n \geq 0$), randomly spread over a fixed area, have been performed. Results are shown in Tables 1a and 1b, where it is reported the improvement with respect to the average map uncertainty and the minimum map uncertainty, respectively (where by "map uncertainty" it is meant the total area of all uncertainty sets in a map). Percentages are evaluated over 100 experiments. The errors on robot absolute orientation measurements have been neglected (which amounts to consider $\epsilon^{v_\theta} \approx 0$), while errors on distance and orientation measurements have been generated as i.u.d. (independent uniformly distributed) signals, satisfying (7) with constant bounds $\epsilon^{v_\alpha} = 3°$ for the latter, while the bounds on the former depend quadratically on the distance measured, i.e. $\epsilon^{v_{d_i}}(k) = \kappa_d \, d_i^2 \, (X(k))$, with $\kappa_d = 0.003$.

| $m \setminus n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 2 | 72.8 | 63.6 | 52.8 | 46 | 45.4 | 43.6 |
| 3 | 74.3 | 71.2 | 67.4 | 63.8 | 64.2 | 57.7 |
| 4 | 77.3 | 74.9 | 72.4 | 70.4 | 69.2 | 67.6 |
| 5 | 79.1 | 77.8 | 76.4 | 74.8 | 72.8 | 72.2 |
| 6 | 80.8 | 80 | 78.9 | 77.6 | 76.9 | 75.8 |

(a)

| $m \setminus n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 2 | 71.9 | 44.5 | 31.6 | 25 | 21.5 | 18.7 |
| 3 | 54.6 | 48.6 | 43.8 | 36.9 | 35.1 | 31.7 |
| 4 | 54.3 | 50.8 | 47.4 | 42.4 | 43.2 | 39.3 |
| 5 | 57.1 | 52.7 | 51.5 | 48.9 | 46.8 | 45 |
| 6 | 57.3 | 57.1 | 55.3 | 53.2 | 51.4 | 48.8 |

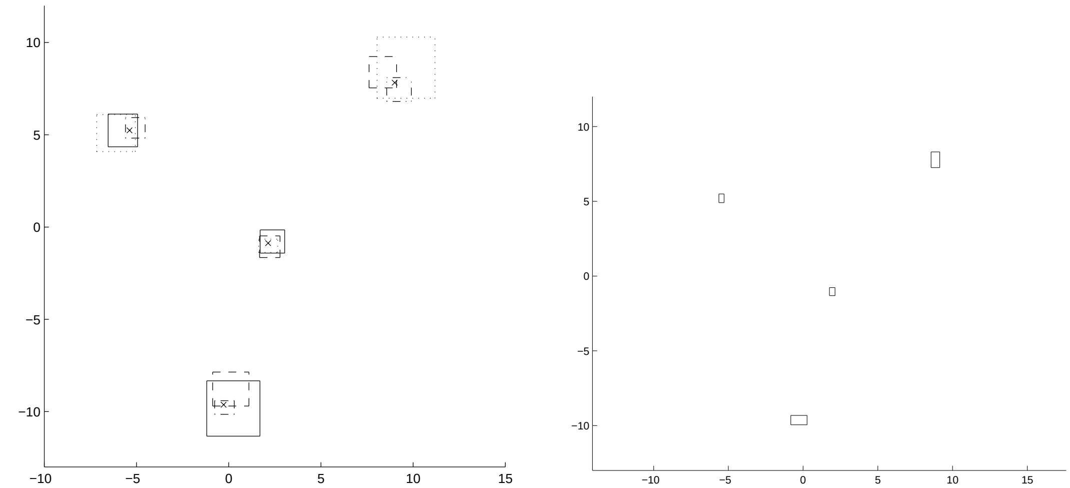
(b)

Table 1. Percentage of total uncertainty reduction due to the fusion algorithm: (a) with respect to the average total uncertainty over all initial maps; (b) with respect to the uncertainty of the best initial map.

The result of a typical run of the map fusion algorithm is shown in Fig. 3, for the case $m = 4$, $n = 0$.

Some remarks can be made by inspection of results in Tables 1a and 1b. Uncertainty reduction provided by the map fusion algorithm becomes more significant when the number $m$ of robots (and consequently of available maps) increases. This happens because more maps will generally provide overall tighter constraints, in the map fusion step. Notice that such improvement occurs also with respect to the best map, and therefore it cannot be attributed to a worse quality of the average map. In addition, for a fixed number of robots, improvements get less remarkable as the number $n$ of non-sensing features increases. As a matter of fact, adding non-sensing features to the environment increases the map uncertainty without providing additional means for uncertainty reduction. Also this trend is confirmed by both tables.

The dynamic multi-robot SM-SLAM algorithm described in Section 3 has been tested in a setting where 4 moving agents and 10 static landmarks are randomly spread over a square region of about 8000 $m^2$. Robots cover rough circles, with different radii. Disturbance $w_i(k)$ on the motion model is generated as




(a) (b)

Fig. 3. Results provided by the set-valued map fusion algorithm. (a): self referenced set-valued map produced by each robot (solid, dashed, dash-dotted and dotted); (b): map produced by the proposed fusion algorithm. All the coordinate systems have been translated in order to have map overlapping.

a nonstationary i.u.d. signal, with mean value proportional to the distance covered during the last robot move. Only one map fusion is performed after the first exteroceptive measurements at time $k = 0$; then the algorithm presented in Section 3 is run. During this experiment, the odometry error bound is set to 10% of the distance covered, uncertainty bound on angle measurements is $\epsilon^{v_\alpha} = 3°$, while the constant for the bound on distance measurements is $\kappa_d = 0.002$. Figures 4a and 4b compare the results provided by the multi-robot SLAM approach to those of the single-robot algorithm. Fig. 4a shows the total area of uncertainty boxes for the 4 robots, computed by each agent via the single-robot SM-SLAM (dashed lines) and by applying the proposed multi-robot SLAM algorithm (solid line). In Fig. 4b the same comparison is depicted for the total area of uncertainty boxes for the 10 landmarks. Total robot uncertainty is reduced by $3 \div 4$ times. Also landmark uncertainty is remarkably smaller, though, as expected, the improvement gets less pronounced as long as more measurements are collected. Note that the average total landmark uncertainty for a single robot SLAM after 40 steps is 1.29 $m^2$, while the average landmark uncertainty for the proposed multirobot approach, after 10 steps, is 1.11 $m^2$, thus showing an improvement of about 15 % with respect to the single robot approach, even on a time horizon four time shorter.

## 6 Conclusions

A set membership approach to the SLAM problem for a team of mobile robots has been proposed. Simulations results demonstrate that the improvement of multi-robot SLAM with respect to the single-robot case is remarkable, both for static map fusion and for dynamic localization and map updating. The computational burden of the proposed algorithm is fairly low, because set approximations are based on the intersection of simple 2D sets (e.g. boxes).

A limitation of the proposed approach concerns correlations between robot and landmark positions. These are neglected in order to simplify set approximation procedures, but they may be included at the price of a higher computational burder. Nevertheless, set membership map fusion presented in Sect. 4 is a possible way to implicitely cope with this problem. Future investigations will concern the challenging problem of multi-robot localization and mapping for a team of indistinguishable agents. This requires a preliminary matching stage, in which the features detected by each robot in its own map are correctly associated to the corresponding features in the other maps. Open problems are how to exploit set-valued maps in order to exclude unfeasible matchings or detect possible ambiguities, as well as the formulation of efficient algorithms that can tackle the matching problem with a reasonable computational effort.

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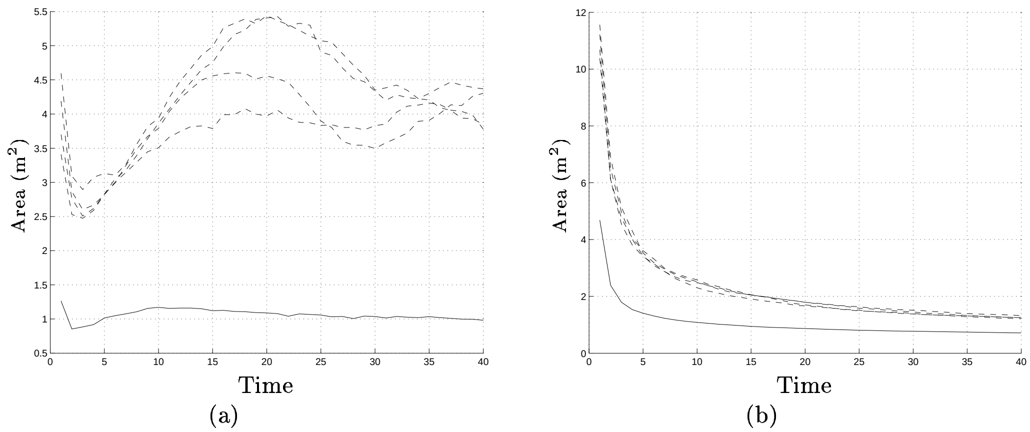


Fig. 4. Comparison of single-robot (dashed lines) and multi-robot (solid line) SM-SLAM: (a) total robot uncertainty; (b) total landmark uncertainty.


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